Pythagorean Theorem Proofs: Connecting Interactive Websites

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Introduction >>

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The Pythagorean Theorem, attributed to Greek mathematician Pythagoras, is one of the most famous and earliest-known theorems to ancient civilizations.

The Babylonians knew about the Pythagorean Theorem about a thousand years before the time of Pythagoras (born in 572 bc) Ancient clay tablets from Babylonia, dated around 1900 bc to 1600 bc, give the earliest tangible record of the Pythagorean Theorem (Neugebauerand Sachs 1945, Neugebauer 1969, Bruins1949, Buck 1980, Friberg 1981). They are known as *Plimpton 322* because they are item number 322 in the G. A. Pilmpton collection at Columbia University. The tablets list columns of numbers showing what we now call *Pythagorean Triples*—sets of numbers that satisfy the theorem. The Babylonian Method found in the Babylonian tablets is as follows:

"4 is the length and 5 the diagonal. What is the breadth? Its size is not known. 4 times 4 is 16. 5 times 5 is 25. You take 16 from 25 and there remains 9. What times what shall I take in order to get 9? 3 times 3 is 9. 3 is the breadth."

Therefore, the general rule is as follows:

"a is the length and c the diagonal. What is the breadth?
Its size is not known.
a times a is a^2.
c times c is c^2.
You take a^2 *from* c^2 *and there remains* b^2.
What times what should I take in order to get b^2?
b times b is b^2.
b is the breadth."

The Chinese used the Pythagorean Theorem as far back as 1000 bc. The theorem went by the name of the *Gougu Theorem*, based on the numerical proof in the Zhoubi Suanjing (*The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*), an ancient Chinese text on astronomy and mathematics, probably dating about 500 BC to 200 BC.

The theorem is taught in schools around the world. Its practical applications have far-reaching ramifications not only in mathematics but also in the arts, science, and other fields. For example, the theorem can be used when determining distances between two points, such as in navigation and land surveying. This theorem is also used frequently in areas such as architecture, construction, and measurement. Pythagoras developed the formula for finding the lengths of the sides of any right

triangle, $a^2 + b^2 = c^2$, where *a* and *b* are the legs of a right triangle and *c* is the hypotenuse.

The Pythagorean Proposition, a book published in 1940, contains 370 different proofs of the Pythagorean theorem, including the one by American President James Garfield. Today there are about four hundred visual, algebraic, and geometric proofs.

The purpose of this paper is to use interactive applets to examine some proofs of the Pythagorean Theorem in different cultures—Greek, Chinese, Hindu, and American. By using the visual and dynamic demonstrations, students can explore numerous geometric and algebraic proofs of the Pythagorean Theorem. They can also learn to analyze and interpret the various proofs of the Pythagorean Theorem.

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A Greek proof >>

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Euclid's proof can be found in Book I of Euclid's *Elements* (Tropfke 1921, Tietze 1965). Euclid showed that the area of rectangle AMLE is equal to the area of square AFGC (see **fig. 1**); and the area of the rectangle MBDL is equal to the area of square CHKB. He used the concept of similarity to prove these equalities.

ΔABF has base AF and altitude FG. Its area, therefore, equals half that of square AFGC.

ΔABF ~ΔAEC by SAS (AE = AB, AF = AC, and m BAF = m BAC + m CAF = m CAB + m BAE = m CAE. Thus, the area of ΔAEC equals half that of the rectangle AFGC.

ΔAEC has base AE and altitude EL. Its area, therefore, equals half that of square AMLE. The area of square AFGC is equal to the area of rectangle AMLE.

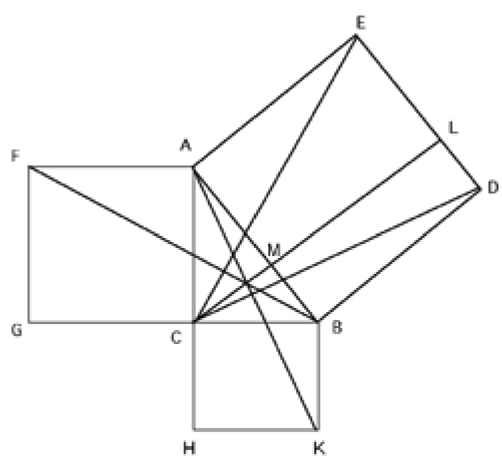

Figure 1 A Greek proof

Similarly, ΔABK has base BK and the altitude HK. Its area equals half that of square CHKB. ΔABK ~ΔBCD by SAS (BD = AB, BC = BK, and m ABK = m ABC + m CBK = m CBA + m ABD = m CBD. Thus, the area of ΔBDC equals half that of rectangle CHKB. ΔBDC has base BD and altitude LD. Its area, therefore, equals half that of square MBDL. Therefore, the area of square BCHK is equal to the area of rectangle MBDL.

The two rectangles AMLE and MBDL make up square ABDE, which is equal in area to square AFGC plus square CHKB.

“An Interactive Proof of Pythagoras’s theorem” at the University of Western Ontario’s Department of Computer Science Web site has a Java applet to illustrate Euclid’s proof (see **fig.2**). This Web site won the grand prize in Sun Microsystem’s Java programming contest in 1995. At this Web site, students see a dynamic demonstration of the Pythagorean relationship. Consider the gray right triangle in the interactive figure. Clicking the *next* button allows students to see that half the sum of the areas of the small squares adds up to half the area of the large square. Change the dimensions of the gray right triangle by clicking anywhere on the picture window. The vertex labeled by the red square will be moved to the mouse position. Click on *next* again. Change the dimensions of the triangle again and repeat the procedure. What do you observe?

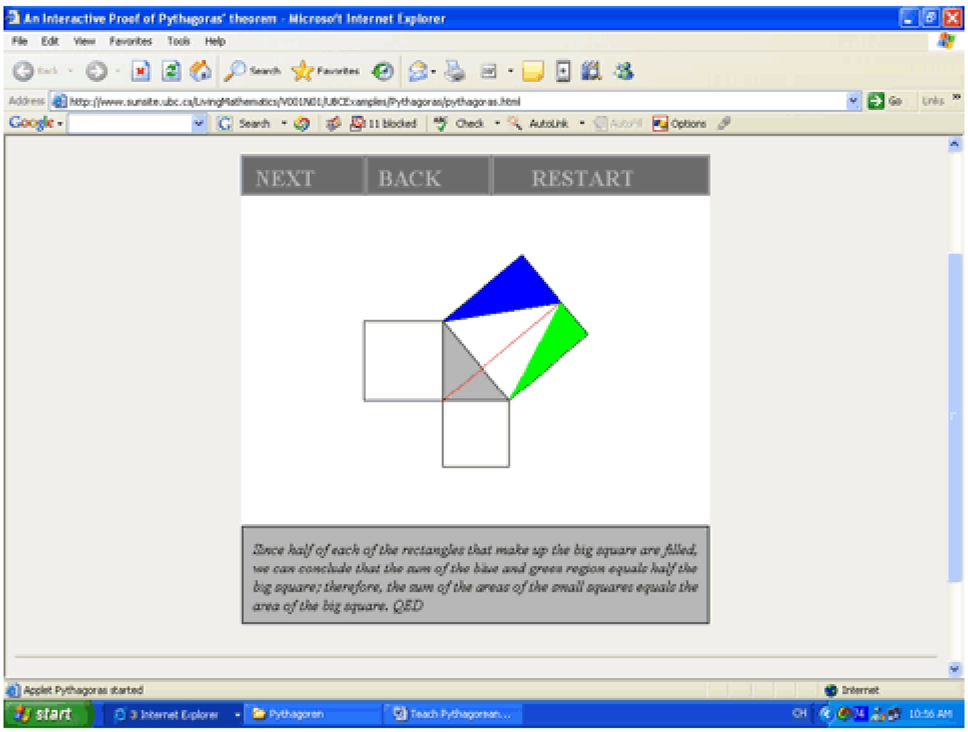

Figure 2 A Greek proof Web site

A Hindu proof >>

The Hindu mathematician Bhaskara (1114-1185) also developed a proof of the Pythagorean Theorem. He drew the diagram shown in **figure 3**. He arranged four red right triangles of side lengths a, b, c into a c-by-c square and filled in the center with the yellow square with side length ab (Gardner 1984). Algebra can be used to verify that this diagram proves the Pythagorean Theorem.

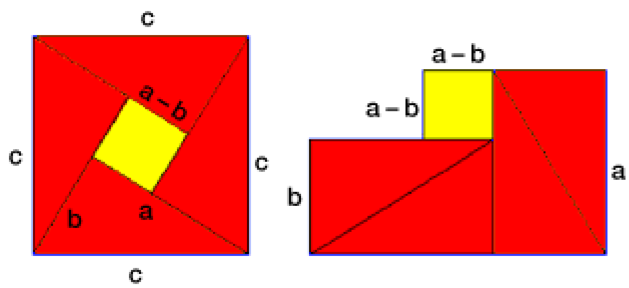

Figure 3 A Hindu proof

The total area of the red triangles is $4 \cdot (\frac{1}{2})\, ab$.
The area of the yellow square is $(a - b)^2$.
The area of the $c \times c$ square is c^2.

Therefore c^2

$$= 4 \cdot (\tfrac{1}{2})\, ab. + (a - b)^2$$
$$= 2ab + a^2 - 2ab + b^2$$
$$= a^2 + b^2$$

Thus, $c^2 = a^2 + b^2$

"Proof without Words: Pythagorean Theorem" at the National Council of Teachers of Mathematics Illuminations Web site provides a step-by step JAVA applet demonstrating the proof that is attributed to Bhaskara (see **fig.4**). The site also has detailed instructions and an exploration of the proof.

At this Web site, you will start arranging copies of the right triangle (Click on *Arrange Four Copies*). Next, click on *Color the Copies*. Click on *Rearrange Shapes*. Click on *Behold*. The solid blue lines indicate how it works. You can start over as you wish by clicking on *Start Over.*

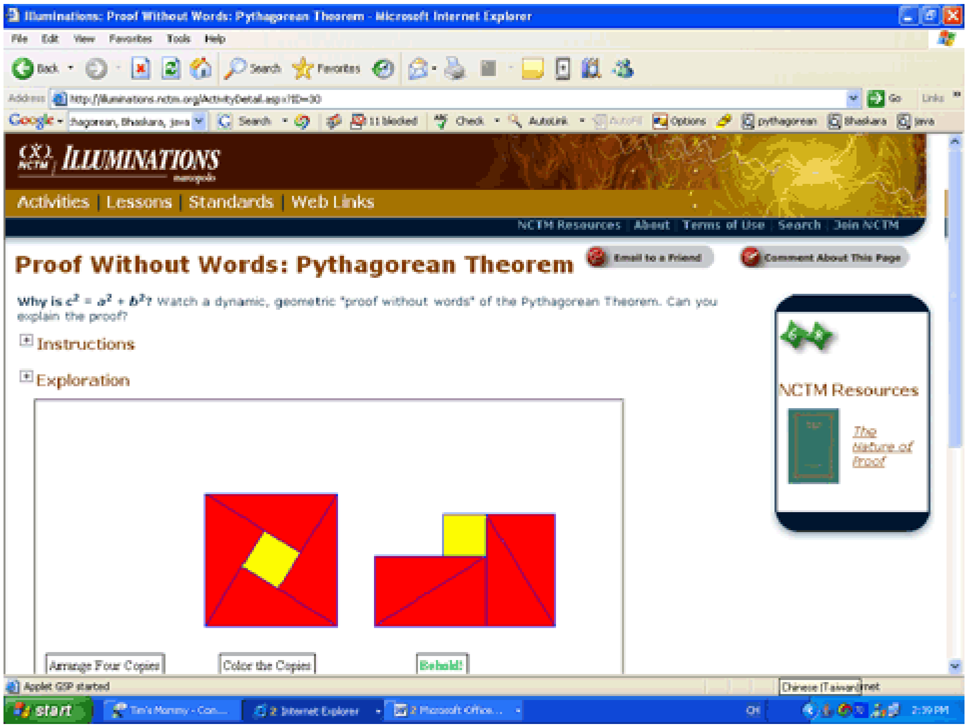

Figure 4 A Hindu proof Web site

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A China proof >>

The *Chou Pei Suan Ching* is one of the earliest mathematical textual sources. The author is unknown. *Chou Pei Suan Ching* gives a visual proof of the Pythagorean theorem for the (3, 4, 5) right triangle. During the Han Dynasty, from 200 BC to AD 200, Pythagorean triples appear in *The Nine Chapters on the Mathematical Art*, together with a mention of right triangles. The visual proof for this theorem is shown in **figure 5**. This proof starts with two congruent squares (Swetz and Kao 1977). Next, rearrange the triangles of the first square to form the rectangles in the second square. Here is the algebraic proof:

Area of first square = $4\left(\frac{1}{2}\right) ab + c^2$

Area of second square $= 4\left(\frac{1}{2}\right) ab + a^2 + b^2$

Area of first square = Area of second square

That is, $4\left(\frac{1}{2}\right) ab + a^2 + b^2 = 4\left(\frac{1}{2}\right) ab + c^2$

Therefore, $a^2 + b^2 = c^2$

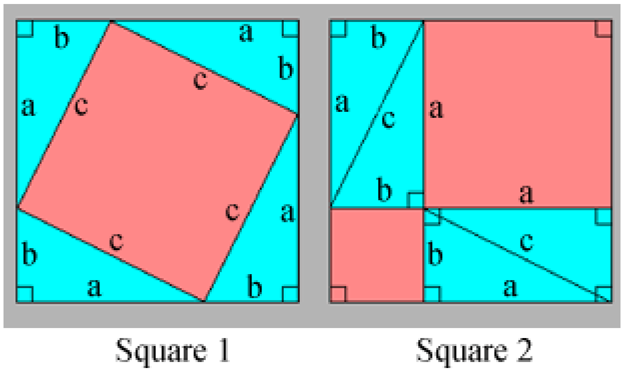

Figure 5 A Chinese proof

"A Picture Proof of the Pythagorean Theorem" on the University of Tennessee at Chattanooga Web site provides a step-by step Java applet to help students understand the Chinese proof as they explore the rearrangement of the triangles (see **fig.6**).

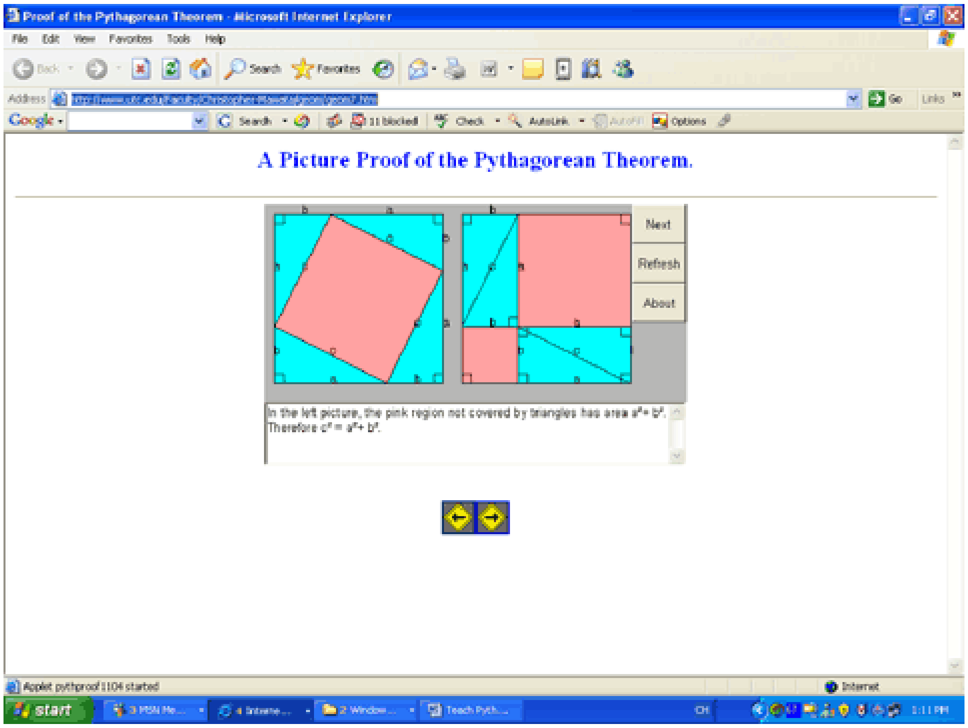

Figure 6 A Chinese proof Web site

An American proof >>

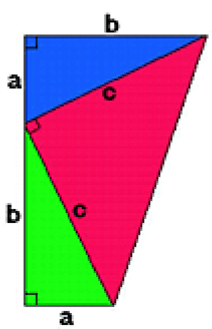

Figure 7 An American proof

In 1876, President James A. Garfield discovered a proof of the Pythagorean Theorem by using the area of a trapezoid:

$A = \frac{1}{2}h(b_1 + b_2)$, where b_1 and b_2 are the bases and h is the altitude. (Gardner 1984; Pappas 1989; Bogomolny 2006).

Prove: The area of the trapezoid is equal to the sum of the areas of the three triangles (see **fig.7**).

Area of trapezoid = area of the blue triangle + area of the pink triangle + area of the green triangle

$$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ba + \frac{1}{2}c^2 + \frac{1}{2}ab$$

$$\frac{1}{2}(a^2 + 2ab + b^2) = \frac{1}{2}ba + \frac{1}{2}c^2 + \frac{1}{2}ab$$

$$\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = \frac{1}{2}ba + \frac{1}{2}c^2 + \frac{1}{2}ab$$

$$\frac{1}{2}a^2 + ab + \frac{1}{2}b^2 = \frac{1}{2}c^2 + ab$$

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{1}{2}c^2$$
$$a^2 + b^2 = c^2$$

The “Garfield’s Proof of the Pythagorean Theorem” Web site at the Colgate University Department of Mathematics has an animated GIF and a step-by-step illustration of President Garfield’s proof (see **fig.8**). Each step of the algebra proof will stay on-screen for about three seconds. The demonstration shows that the area of the trapezoid is equal to the sum of the areas of the blue, red, and green triangles.

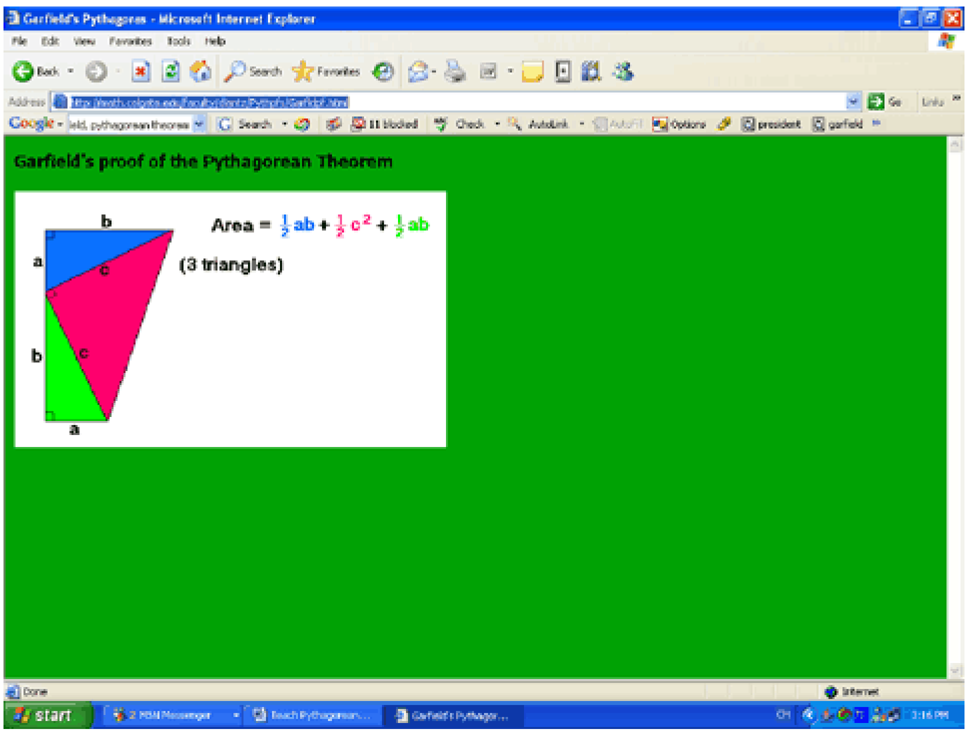

Figure 8 An American proof Web site

Connecting to Teaching Using Interactive Web Sites >>

According to NCTM *Principles and Standards for School Mathematics*, “Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students’ learning” (NCTM 2000, p. 11). Interactive Web sites can help students to explore dynamic demonstrations of the Pythagorean relationship. In addition, visual and dynamic demonstrations can help students to analyze and explain mathematical relationships.

Click here for an example of a lesson plan that provides teachers with an idea of how to use interactive Web sites of Pythagorean Theorem Proofs in the classroom.

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Conclusion >>

A proof is a valid argument, using deductive reasoning. Interactive Web sites present proofs to help students better understand the process of deductive reasoning and the Pythagorean Theorem. Therefore, encouraging students to use the Internet to investigate and prove the Pythagorean Theorem is essential to aid students in recognizing, appreciating, and using applications of the Pythagorean Theorem in both algebraic and geometric terms. They also ought to be encouraged to explain the theorem in their own words to move them to higher-order thinking.

The Pythagorean Theorem is about 2500 years old (Veljan2000). We appreciate the work of all the mathematicians who developed the large number and variety of proofs that help us see how mathematicians pose a problem and establish its validity. The Pythagorean Theorem can be proved using deductive reasoning, the process of reasoning logically from given statements to a conclusion. (Euclid's proof is a deductive proof sequence in geometry.) Teaching proofs to students and encouraging them to develop their own proofs of the Pythagorean Theorem help them learn to think logically and strengthen their deductive reasoning. The Chinese and Hindu proofs are nice examples of visual proofs. By rearranging triangles on the interactive Web sites, students can easily grasp the concepts behind these proofs. The Greek proof is one of the geometric proofs. The demonstration of congruent triangles shown at the interactive Web sites allows students to see the proof without words.

Some possible extensions of this lesson include the following:

- If the Pythagorean Theorem is true, consider $a^n + b^n = c^n$, $n > 2$. Does this formula work just like the Pythagorean Theorem? Can you find an interactive Web site with a proof of this theorem?
- Consider the converse of the Pythagorean Theorem and generalize the theorem to shapes other than triangles.
- Can you find the Pythagorean Theorem in everyday life? What real world applications of Pythagorean Theorem are there?

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An example of a lesson plan using interactive Web sites

Title: Explore Pythagorean Theorem Proofs

Grade Level:7–12

Objectives:

1. Discover the Pythagorean Theorem with the aid of animated applets.
2. Explore methods of proving the Pythagorean Theorem.
3. Analyze and explain mathematical relationships of various proofs of the Pythagorean Theorem.

Materials:

1. Computers and internet access
2. White board or chalkboard
3. Board markers or chalk
4. Pencils and paper

Lesson:

1. Briefly explain that we are going to be discussing the Pythagorean Theorem.
2. Distribute any teacher made handouts and give students direction.
3. Have students research Greek, Hindu, Chinese, and American proofs of the Pythagorean Theorem.
4. When students have completed their research, ask them to summarize their findings in a one-page report.
5. Ask students to explore animated applets of the Pythagorean Theorem proof.
6. Ask students to write a proof outline showing how the geometric constructions imply the Pythagorean Theorem.
7. Ask students to demonstrate a proof of the Pythagorean Theorem.
8. Have students analyze and explain the mathematical relationships of in the various proofs of the Pythagorean Theorem.

Homework Assignments:

1. What is your favorite proof of the Pythagorean Theorem? Explain why.
2. Select a favorite proof of the Pythagorean Theorem and create a colored illustration of the proof.
3. Which animated applets help you better understand the proof of the Pythagorean Theorem? Explain why.
4. Find more animated applets of the proof of the Pythagorean Theorem based on algebraic proof by exploring the Internet?
5. Can you find more animated applets of the proof of the Pythagorean Theorem based on geometric proofs?
6. Which proofs are based on similarity, rotations, or translations? Explain your answer.

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